REDUCTION OF A LINEAR DYNAMICAL SYSTEM AND THE PROBLEM OF MOMENTS

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Abstract: This contribution is about the connection between the problem of moments and model reduction in linear dynamical systems. It is given a short introduction about the problem of moments and some of many connections one can find between different methods with its help are shown. Especially it is described that the minimal partial realization introduced by R. E. Kalman in 1979 is closely connected with the truncated problem of moments.

Keywords: dynamical system, moment, transfer function

1 INTRODUCTION

This contribution is about the connection between different methods in mathematics and control theory. The problem of moments can be seen as the central point for these connections. During the last 150 years many books and papers have been published about this problem. Many mathematicians studied it from many different points of view. It is very interesting how many connections between the different parts of mathematics has been found in these works. One can see the classical references [13] and [2]. An interesting historical review about the birth of the problem of moments can be found in [9]. As the time went on, the problem of moments was used in order to solve various questions in mathematical statistics, theory of probability and mathematical analysis.

2 FORMULATION OF THE PROBLEM OF MOMENTS

Given the sequence of real numbers $\{\xi_k\}_{k=0}^{\infty}$. The problem is to find the following positive measure μ such that

$$\xi_k = \int_I \lambda^k d\mu(\lambda), \qquad k = 0, 1, \dots. \tag{1}$$

In the case when $I = [0, \infty)$ we talk about the Stieltjes moment problem. The case when $I = \mathbb{R}$ is called the Hamburger moment problem. The real numbers $\{\xi_k\}_{k=0}^{\infty}$ are then called the moments. The terminology was taken from mechanics. If the measure μ represents the distribution of the mass over the real semi-axis, then the integrals

$$\int_0^{\infty} \lambda d\mu(\lambda), \int_0^{\infty} \lambda^2 d\mu(\lambda)$$

represent the first (statical) moment and the second moment (moment of inertia).

One can ask the following questions:

- Does the measure μ exist for the sequence of the moments $\{\xi_k\}_{k=0}^{\infty}$?
- If the measure μ exists, is it determined uniquely?

The moments can be also expressed in the matrix language. Let's consider an HPD (Hermitian positive definite) matrix $A \in \mathbb{C}^{N\times N}$ and the vector v_1. Since A is HPD we can do its spectral decomposition

$$AU = U\Lambda,$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}, \quad U = [u_1, u_2, ..., u_N], \quad U^*U = UU^* = I,$$

where $u_1, u_2, ..., u_N$ are the column vectors of the matrix U. Now consider the non-decreasing measure $\mu(\lambda)$ with the points of increase equal to the eigenvalues of A and the weights μ_l equal to sizes of the squared components of v_1 in the corresponding invariant subspaces, i.e.

$$\mu_l = |(v_1, u_l)|^2, \quad l = 1, ..., N.$$

With the above setting and with the assumption that all eigenvalues of A lie inside interval I the moments can be expressed as

$$\xi_k = \int_I \lambda^k d\mu(\lambda) = \sum_{l=1}^{N} \mu_l \{\lambda_l\}^k = v_1^* A^k v_1, \quad k = 0, 1, ...,$$

Let's take a look on the similar problem. Given the same sequence of the moments $\{\xi_k\}_{k=0}^{\infty}$. The following positive measure μ_n is sought such that the first $2n$ moments are matched, i.e.,

$$\xi_k = \int_I \lambda^k d\mu_n(\lambda) = \sum_{l=1}^{n} \mu_l^{(n)} \{\lambda_l^{(n)}\}^k, \quad k = 0, 1, ...2n-1. \tag{2}$$

The formulation above is often called the truncated problem of moments, one can see e.g. [1].

It is known for a long time that the finding of the μ_n instead of μ is closely connected with the Gauss-Christoffel quadrature, see e.g. [16], [10]. Under certain settings the problem of moments can be seen as the theoretical background for the Lanczos method and the CG method. The connection with the CG and with the the Gauss-Christoffel quadrature is known since the introduction of the CG and it was well described by M. R. Hestenes and E. Stiefel in their joint paper [7].

The problem of moments is also closely connected with the Sturm-Liouville problem. In [4] the connections between the singular Sturm-Liouville problem, Jacobi matrices and Hamburger moment problem are described in an elegant way. The nature of the solutions of the singular Sturm-Liouville problem is connected with the determinacy of the associated Hamburger moment problem.

In [11] the results about the sensitivity of the Gauss-Christoffel quadrature with respect to the small perturbations of the measure are given. Obtaining of these results would not be possible without the deep knowledge of the connection of Gauss-Christoffel quadrature with the problem of moments.

3 CONNECTION WITH THE MODEL REDUCTION OF LARGE-SCALE DYNAMICAL SYSTEMS

There is the connection between the model reduction in the linear dynamical systems and the problem of moments. Consider the following dynamical system

$$\begin{aligned} x'(t) &= Ax(t) + bu(t), \\ y(t) &= b^* x(t), \end{aligned} \tag{3}$$

where $A \in \mathbb{C}^{N\times N}$ is the HPD matrix, $b \in \mathbb{C}^N$ and the initial condition is $x(0) = 0$. In [10, pp. 101-108] an elegant description of the connection between the model reduction of the above system and the problem of moments is given. Applying the Laplace transform

$$F(\lambda) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-\lambda t}dt$$

the system (3) can be represented by the transfer function description

$$T(\lambda) = \frac{Y(\lambda)}{U(\lambda)} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}}.$$

If we apply the Laplace transform on the first equation of the system (3) we will get

$$\lambda X(\lambda) = AX(\lambda) + bU(\lambda),$$

so

$$X(\lambda) = (\lambda I - A)^{-1}bU(\lambda). \tag{4}$$

If we apply the Laplace transform on the second equation of the system (3) we will get

$$Y(\lambda) = b^*X(\lambda) = b^*(\lambda I - A)^{-1}bU(\lambda). \tag{5}$$

after substitution from (4) to (5) we will get the transfer function of (3) in the following form

$$T(\lambda) = \frac{Y(\lambda)}{U(\lambda)} = b^*(\lambda I - A)^{-1}b, \quad \lambda \in \mathbb{C}. \tag{6}$$

The model reduction problem is to find the reduced order A_n, b_n such that $T_n(\lambda)$ which is defined as follows

$$T_n(\lambda) = b_n^*(\lambda I - A_n)^{-1}b_n, \quad \lambda \in \mathbb{C},$$

approximates in some sense well $T(\lambda)$ for λ from some subset of $\mathbb{C}$. The double (A_n, b_n) is called a realization of T. In [12, p. 9] it is shown, that in certain cases it is possible to have a double (A_n, b_n) with $A_n \in \mathbb{C}^{n\times n}$ and $n < N$ which correspond to the same transfer function T as does (A, b). Such a realization with additional condition that A_n has minimal dimension is called a minimal realization. The problem of finding efficient numerical approximation to (6) arises in many applications unrelated to linear dynamical systems (3). A more general case can be written as

$$c^*F(A)b,$$

where $F(A)$ is a given function of the matrix A and $c \in \mathbb{C}^N$. The particular case $c = b$ and

$$F(A) = (\lambda I - A)^{-1},$$

i.e., $F(A)$ is equal to the matrix resolvent, where λ is outside of the spectrum A, is of great importance.

At first consider the expansion of $T(\lambda)$ about infinity

$$\begin{aligned} T(\lambda) &= \lambda^{-1}b^*(I - \lambda^{-1}A)^{-1}b = \\ &= \lambda^{-1}(b^*b) + \lambda^{-2}(b^*Ab) + \ldots + \lambda^{-2n}(b^*A^{2n-1}b) + \ldots. \end{aligned}$$

A reduced model of order n which matches the first $2n$ terms in the above expansion is known as the minimal partial realization. The concept of the minimal partial realization was introduced in the control theory literature by R. E. Kalman in 1979, see [8]. The idea to find the reduced model is nothing else than the problem of moments such that the first $2n$ moments are matched, see (2).

Consider the measure μ with N points of increase associated with the HPD matrix A and the initial vector b. Then

$$b^*(\lambda I - A)^{-1} b = \sum_{j=1}^{N} \frac{\mu_j}{\lambda - \lambda_j} = \mathcal{F}_N(\lambda),$$

where the continued fraction $\mathcal{F}_N(\lambda)$ can be for any $n < N$ expanded to

$$\mathcal{F}_N(\lambda) = \sum_{k=1}^{2n} \frac{\xi_{k-1}}{\lambda^k} + O\left(\frac{1}{\lambda^{2n+1}}\right) = \mathcal{F}_n(\lambda) + O\left(\frac{1}{\lambda^{2n+1}}\right),$$

where ξ_{k-1} are defined as follows

$$\xi_{k-1} = \int_I \lambda^{k-1} d\mu(\lambda) = \sum_{l=1}^{n} \mu_l^{(n)} \{\lambda_l^{(n)}\}^{k-1}, \qquad k = 1, 2, ..., 2n.$$

ξ_{k-1} can be also written in the matrix form

$$\xi_{k-1} = b^* A^{k-1} b.$$

$\mathcal{F}_n(\lambda)$ approximates $\mathcal{F}_N(\lambda)$ with the error proportional to $1/\lambda^{2n+1}$. The minimal partial realization in model reduction of linear dynamical systems matches the first $2n$ moments

$$\xi_{k-1} = b^* A^{k-1} b, \qquad k = 1, ..., 2n.$$

Sometimes it is more convenient to do the model reduction with the expansion of the $T(\lambda)$ in the neighborhood of some $\lambda_0 \in \mathbb{C}$, see e.g., [3]. Here, the case where $\lambda_0 = 0$ is used. The model reduction is achieved by matching the moments

$$\xi_k = b^*(A^{-1})^k b, \qquad k = 1, 2, ...$$

of the expansion

$$\begin{aligned} -T(\lambda) &= b^* A^{-1}(I - \lambda A^{-1})^{-1} b = \\ &= b^* A^{-1} b + \lambda(b^* A^{-2} b) + ... + \lambda^{2n-1}(b^* A^{-2n} b) + ... \end{aligned}$$

An interesting case is when we take $\lambda = 0$ and the vector $c \in \mathbb{C}^N$ instead of the first vector $b \in \mathbb{C}^N$ in (6). We will get the following quantity

$$c^* A^{-1} b.$$

This quantity is called the scattering amplitude in control theory literature. Its approximation is very important in many applications, see e.g., [5] and [6]. However as pointed out in [15] the problem of numerical approximation of the single scalar value $c^* A^{-1} b$ is different from the numerical approximation of the whole transfer function $T(\lambda)$ and therefore a different approach must be taken. The approach to approximate this quadratic form $c^* A^{-1} b$ was taken in the paper [14] and the Vorobyev moment problem (see [17]) was used in order to get the good approximation.

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